The Data Cube as a Typed Linear Algebra Operator

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Motivation

"Only by taking infinitesimally small units for observation (the ***differential*** *of history, that is, the individual tendencies of men) and attaining to the art of* ***integrating*** *them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history."*

Leo Tolstoy, "War and Peace"
- Book XI, Chap.II (1869)

150 years later, this is what we are trying to attain through **data-mining**.

But — how fit are our **maths** for the task?

Have we attained the "**art of integration**"?

Motivation

Since the early days of psychometrics in the **social sciences** (1970s), **linear algebra** (LA) has been central to data analysis (e.g. tensor decompositions etc)

We follow this trend but in a **typed** way, merging **LA** with polymorphic **type systems**, over a categorial basis.

We address a concrete example: that of studying the maths behind a well-known device in data analysis, the **data cube** construction.

We will define this construction as a **polymorphic** LA operator.

Typed **linear algebra** is proposed as a rich setting for such an "**art of integration**" to be achieved.

Running example

Raw data:

$t =$

#	**Model**	**Year**	**Color**	**Sale**
1	*Chevy*	1990	*Red*	5
2	*Chevy*	1990	*Blue*	87
3	*Ford*	1990	*Green*	64
4	*Ford*	1990	*Blue*	99
5	*Ford*	1991	*Red*	8
6	*Ford*	1991	*Blue*	7

Columns — attributes — the *observables*
Rows — records (n-many) — the *infinitesimals*

Column-orientation — each column (attribute) A represented by a function $t_A : n \to A$ such that $a = t_A\ (i)$ means "a is the value of attribute A in record nr i".

Records are tuples

Can records be rebuilt from such **attribute projection** functions?

Yes — by **tupling** them.

Tupling: *Given functions $f : A \to B$ and $g : A \to C$, their tupling is the function $f \vartriangledown g$ such that*

$$(f \vartriangledown g)\ a = (f\ a, g\ a)$$

For instance,

$$(t_{Color} \vartriangledown t_{Model})\ 2 = (Blue, Chevy),$$
$$(t_{Year} \vartriangledown (t_{Color} \vartriangledown t_{Model}))\ 3 = (1990, (Green, Ford))$$

and so on.

Inverting tuples

For the column-oriented model to work one will need to express *joins*, and these call for *"inverse"* functions, e.g.

$$(t_{Model} \triangledown t_{Year})^{\circ}\ (Ford, 1990) = \{3, 4\}$$

meaning that tuples nr 3 and nr 4 have the same model (*Ford*) and year (1990).

However, the type $f^{\circ} : A \rightarrow \mathcal{P}\ n$ is rather annoying, as it involves **sets** of tuple indices — these will add an extra layer of complexity.

Fortunately, there is a simpler way — **typed linear algebra**, also known as **linear algebra of programming** (**LAoP**).

The LAoP approach

Represent functions by Boolean matrices.

Given (finite) types A and B, any function

$$f : A \to B$$

can be represented by a matrix $[\![f]\!]$ with A-many columns and B-many rows such that, for any $b \in B$ and $a \in A$, matrix cell

$$b\ [\![f]\!]\ a = \begin{cases} 1 \Leftarrow b = f\ a \\ 0\ \mathit{otherwise} \end{cases}$$

NB: Following the **infix** notation usually adopted for relations (which are Boolean matrices) — for instance $y \leqslant x$ — we write $y\ M\ x$ to denote the contents of the cell in matrix M addressed by row y and column x.

The LAoP approach

One projection function (matrix) per **dimension** attribute:

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

t_{Year}	1	2	3	4	5	6
1990	1	1	1	1	0	0
1991	0	0	0	0	1	1

t_{Color}	1	2	3	4	5	6
Blue	0	1	0	1	0	1
Green	0	0	1	0	0	0
Red	1	0	0	0	1	0

#	**Model**	**Year**	**Color**	**Sale**
1	Chevy	1990	Red	5
2	Chevy	1990	Blue	87
3	Ford	1990	Green	64
4	Ford	1990	Blue	99
5	Ford	1991	Red	8
6	Ford	1991	Blue	7

NB: we tend to abbreviate $[\![f]\!]$ by f when the context is clear.

The LAoP approach

Note how the inverse of a function is also represented by a Boolean matrix, e.g.

t°_{Model}	Chevy	Ford
1	1	0
2	1	0
3	0	1
4	0	1
5	0	1
6	0	1

versus

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

— no need for powersets.

Clearly,

$$j \ t^{\circ}_{Model} \ a = a \ t_{Model} \ j$$

Given a matrix M, M° is known as the **transposition** of M.

The LAoP approach

We **type** matrices in the same way as functions: $M : A \to B$ means a matrix M with A-many columns and B-many rows.

Matrices are arrows: $A \xrightarrow{M} B$ denotes a matrix from A (source) to B (target), where A, B are (finite) types.

Writing $B \xleftarrow{M} A$ means the same as $A \xrightarrow{M} B$.

Composition — *aka* matrix multiplication:

$$
\begin{tikzcd}
B & A \arrow[l, "M"'] & C \arrow[l, "N"'] \arrow[ll, bend left, "M \cdot N"]
\end{tikzcd}
$$

$$b(M \cdot N)c \;=\; \langle \sum a \;::\; (b\ M\ a) \times (a\ N\ c) \rangle$$

The LAoP approach

Function composition implemented by matrix multiplication, $[\![f \cdot g]\!] = [\![f]\!] \cdot [\![g]\!]$

Identity — the identity matrix id corresponds to the identity function and is such that

$$M \cdot id = M = id \cdot M \tag{1}$$

Function tupling corresponds to the so-called **Khatri-Rao product** $M \triangledown N$ defined index-wise by

$$(b, c)\ (M \triangledown N)\ a \quad = \quad (b\ M\ a) \times (c\ N\ a) \tag{2}$$

Khatri-Rao is a "column-wise" version of the well-known *Kronecker product* $M \otimes N$:

$$(y, x)\ (M \otimes N)\ (b, a) \quad = \quad (y\ M\ b) \times (x\ N\ a) \tag{3}$$

Typing data

The raw data given above is represented in the LAoP by the expression

$$v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ}$$

of type

$$v : 1 \rightarrow (Year \times (Color \times Model))$$

depicted aside.

Year x (Color x Model)			ALL
1990	Blue	Chevy	87
		Ford	99
	Green	Chevy	0
		Ford	64
	Red	Chevy	5
		Ford	0
1991	Blue	Chevy	0
		Ford	7
	Green	Chevy	0
		Ford	0
	Red	Chevy	0
		Ford	8

v is a **multi-dimensional** column vector — a **tensor**. Datatype $1 = \{\mathrm{ALL}\}$ is the so-called **singleton** type.

Dimensions and measures

Sale is a special kind of data — a **measure**. Measures are encoded as **row** vectors, e.g.

t^{Sale}	*1*	*2*	*3*	*4*	*5*	*6*
1	*5*	*87*	*64*	*99*	*8*	*7*

recall

#	**Model**	**Year**	**Color**	**Sale**
1	*Chevy*	*1990*	*Red*	*5*
2	*Chevy*	*1990*	*Blue*	*87*
3	*Ford*	*1990*	*Green*	*64*
4	*Ford*	*1990*	*Blue*	*99*
5	*Ford*	*1991*	*Red*	*8*
6	*Ford*	*1991*	*Blue*	*7*

$$\begin{tikzcd} & Model & \\ Year & \#t \arrow[l, "t_{Year}"'] \arrow[r, "t_{Color}"] \arrow[u, "t_{Model}"'] \arrow[d, "t^{Sale}"] & Color \\ & 1 & \end{tikzcd}$$

Summary: **dimensions** *are* **matrices**, **measures** *are* **vectors**.

Measures provide for **integration** in Tolstoy's sense — *aka* **consolidation**

Totalisers

There is a unique function in type $A \to 1$, usually named $A \xrightarrow{!} 1$. This corresponds to a row vector wholly filled with 1s.

Example: $2 \xrightarrow{!} 1 \; = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Given $M : B \to A$, the expression $! \cdot M$ (where $A \xrightarrow{!} 1$) is the row vector (of type $B \to 1$) that contains all column **totals** of M,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 50 & 40 & 85 & 115 \\ 50 & 10 & 85 & 75 \end{bmatrix} = \begin{bmatrix} 100 & 50 & 170 & 190 \end{bmatrix}$$

Given type A, define its **totalizer** matrix $A \xrightarrow{\tau_A} A + 1$ by

$$\begin{aligned} \tau_A &: A \to A + 1 \\ \tau_A &= \begin{bmatrix} id \\ \hline ! \end{bmatrix} \end{aligned} \tag{5}$$

Thus $\tau_A \cdot M$ yields a copy of M on top of the corresponding totals.

Cubes

Data **cubes** can be obtained from products of totalizers.

Recall the Kronecker (tensor) product $M \otimes N$ of two matrices $A \xrightarrow{M} B$ and $C \xrightarrow{N} D$, which is of type $A \times C \xrightarrow{M \otimes N} B \times D$.

The matrix

$$A \times B \xrightarrow{\tau_A \otimes \tau_B} (A+1) \times (B+1)$$

provides for totalization on the **two dimensions** A and B.

Indeed, type $(A+1) \times (B+1)$ is isomorphic to $A \times B + A + B + 1$, whose four parcels represent the four elements of the "**dimension powerset** of $\{A, B\}$".

Cube = muti-dimensional totalisation

Recalling

$$v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ}$$

build

$$c = (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v$$

This is the multidimensional vector (tensor) representing the **data cube** for

- **dimensions** *Year*, *Color*, *Model*
- **measure** *Sale*

depicted aside.

(Year+1) x ((Color+1) x (Model+1))			ALL
1990	Blue	Chevy	87
		Ford	99
		ALL	186
	Green	Chevy	0
		Ford	64
		ALL	64
	Red	Chevy	5
		Ford	0
		ALL	5
	ALL	Chevy	92
		Ford	163
		ALL	255
1991	Blue	Chevy	0
		Ford	7
		ALL	7
	Green	Chevy	0
		Ford	0
		ALL	0
	Red	Chevy	0
		Ford	8
		ALL	8
	ALL	Chevy	0
		Ford	15
		ALL	15
ALL	Blue	Chevy	87
		Ford	106
		ALL	193
	Green	Chevy	0
		Ford	64
		ALL	64
	Red	Chevy	5
		Ford	8
		ALL	13
	ALL	Chevy	92
		Ford	178
		ALL	270

Totalisers yield cubes

We reason:

$$
\begin{array}{cl}
c & = (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v \\
= & \qquad \{\ v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ}\ \} \\
 & (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ} \\
= & \qquad \{\ \text{property } (M \otimes N) \cdot (P \triangledown Q) = (M \cdot P) \triangledown (N \cdot Q)\ \} \\
 & ((\tau_{Year} \cdot t_{Year}) \triangledown ((\tau_{Color} \cdot t_{Color}) \triangledown ((\tau_{Model} \cdot t_{Model})))) \cdot (t^{Sale})^{\circ} \\
= & \qquad \{\ \text{define } t'_A = \tau_A \cdot t_A\ \} \\
 & (t'_{Year} \triangledown (t'_{Color} \triangledown t'_{Model})) \cdot (t^{Sale})^{\circ}
\end{array}
$$

Note that $t'_A = \left[\frac{t_A}{!}\right]$, since t_A is a function.

Generalizing data cubes

In our approach a **cube** is not necessarily one such column vector.

The key to generic data cubes is (generalized) **vectorization**, a kind of "**matrix currying**": given $A \times B \xrightarrow{M} C$ with $A \times B$-many columns and C-many rows, reshape M into its **vectorized** version $B \xrightarrow{\mathbf{vec}_A\, M} A \times C$ with B-many columns and $A \times C$-many rows.

Such matrices, M and $\mathbf{vec}_A\, M$, are **isomorphic** in the sense that they contain the same information in different formats, as

$$c\ M\ (a, b) \quad = \quad (a, c)\ (\mathbf{vec}_A\ M)\ b \tag{6}$$

holds for every a, b, c.

Generalizing data cubes

Vectorization thus has an inverse operation — **unvectorization**:

$$\begin{tikzcd} A \times B \to C \arrow[rr, bend left, "\mathbf{vec}_A"] \arrow[rr, phantom, "\cong"] & & B \to A \times C \arrow[ll, bend left, "\mathbf{unvec}_A"] \end{tikzcd}$$

That is, M can be retrieved back from $\mathbf{vec}_A\, M$ by devectorizing it:

$$N = \mathbf{vec}_A\, M \quad \Leftrightarrow \quad \mathbf{unvec}_A\, N = M \tag{7}$$

Vectorization has a rich algebra, e.g. a **fusion**-law

$$(\mathbf{vec}\, M) \cdot N \quad = \quad \mathbf{vec}\,(M \cdot (id \otimes N)) \tag{8}$$

and an **absorption**-law:

$$\mathbf{vec}\,(M \cdot N) \quad = \quad (id \otimes M) \cdot \mathbf{vec}\, N \tag{9}$$

(De)vectorization

Devectorizing our starting tensor, across dimension *Year*:

$$\mathbf{unvec}_{Year} \begin{array}{c} Year \times (Color \times Model) \longleftarrow 1 \\ \left(\begin{array}{ccc|c} & & & \text{ALL} \\ \hline 1990 & Blue & Chevy & 87 \\ & & Ford & 99 \\ & Green & Chevy & 0 \\ & & Ford & 64 \\ & Red & Chevy & 5 \\ & & Ford & 0 \\ 1991 & Blue & Chevy & 0 \\ & & Ford & 7 \\ & Green & Chevy & 0 \\ & & Ford & 0 \\ & Red & Chevy & 0 \\ & & Ford & 8 \end{array}\right) \end{array} = \begin{array}{c} Color \times Model \longleftarrow Year \\ \begin{array}{cc|cc} & & 1990 & 1991 \\ \hline Blue & Chevy & 87 & 0 \\ & Ford & 99 & 7 \\ Green & Chevy & 0 & 0 \\ & Ford & 64 & 0 \\ Red & Chevy & 5 & 0 \\ & Ford & 0 & 8 \end{array} \end{array}$$

There is room for further devectorizing the outcome, this time across *Color* — next slide:

(De)vectorization

Further devectorization:

$$\mathit{Color} \times \mathit{Model} \longleftarrow \mathit{Year} \qquad\qquad \mathit{Model} \longleftarrow \mathit{Color} \times \mathit{Year}$$

$$\textbf{\textit{unvec}}_{\mathit{Color}} \left(\begin{array}{ll|cc} & & 1990 & 1991 \\ \hline \mathit{Blue} & \mathit{Chevy} & 87 & 0 \\ & \mathit{Ford} & 99 & 7 \\ \hline \mathit{Green} & \mathit{Chevy} & 0 & 0 \\ & \mathit{Ford} & 64 & 0 \\ \hline \mathit{Red} & \mathit{Chevy} & 5 & 0 \\ & \mathit{Ford} & 0 & 8 \end{array} \right) =$$

$$\begin{array}{l|cc|cc|cc} & \multicolumn{2}{c}{\mathit{Blue}} & \multicolumn{2}{c}{\mathit{Green}} & \multicolumn{2}{c}{\mathit{Red}} \\ \cline{2-7} & 1990 & 1991 & 1990 & 1991 & 1990 & 1991 \\ \hline \mathit{Chevy} & 87 & 0 & 0 & 0 & 5 & 0 \\ \mathit{Ford} & 99 & 7 & 64 & 0 & 0 & 8 \end{array}$$

and so on.

Generic cubes

It turns out **that** cubes can be calculated for any such two-dimensional versions of our original data tensor, for instance,

$$\mathbf{cube}\ N : \ Model + 1 \longleftarrow (Color + 1) \times (Year + 1)$$
$$\mathbf{cube}\ N = \tau_{Model} \cdot N \cdot (\tau_{Color} \otimes \tau_{Year})^{\circ}$$

where N stands for the second matrix of the previous slide, yielding

	Blue			*Green*			*Red*			ALL		
	1990	1991	ALL	1990	1991	ALL	1990	1991	ALL	1990	1991	ALL
Chevy	87	0	87	0	0	0	5	0	5	92	0	92
Ford	99	7	106	64	0	64	0	8	8	163	15	178
ALL	186	7	193	64	0	64	5	8	13	255	15	270

See how the 36 entries of the original cube have been rearranged in a 3*12 rectangular layout, as dictated by the **dimension** cardinalities.

The **cube** (LA) operator

Definition (Cube)

Let M be a matrix of type

$$\Pi_{j=1}^{n} B_j \xleftarrow{M} \Pi_{i=1}^{m} A_i \qquad (10)$$

We define matrix **cube** M, the *cube of* M, as follows

$$\textbf{cube}\ M = \Big(\bigotimes_{j=1}^{n} \tau_{B_j}\Big) \cdot M \cdot \Big(\bigotimes_{i=1}^{m} \tau_{A_i}\Big)^{\circ} \qquad (11)$$

where $\bigotimes$ is finite Kronecker product.

So **cube** M has type $\Pi_{j=1}^{n}(B_j + 1) \longleftarrow \Pi_{i=1}^{m}(A_i + 1)$.

□

Properties of data cubing

Linearity:

$$\textbf{cube}\ (M + N) \quad = \quad \textbf{cube}\ M + \textbf{cube}\ N \tag{12}$$

Proof: Immediate by bilinearity of matrix composition:

$$M \cdot (N + P) \quad = \quad M \cdot N + M \cdot P \tag{13}$$
$$(N + P) \cdot M \quad = \quad N \cdot M + P \cdot M \tag{14}$$

This can be taken advantage of not only in **incremental** data cube construction but also in **parallelizing** data cube generation.

Properties of data cubing

Updatability: by Khatri-Rao product linearity,

$$(M + N) \triangledown P = M \triangledown P + N \triangledown P$$
$$P \triangledown (M + N) = P \triangledown M + P \triangledown N$$

the **cube** operator commutes with the usual CRUDE operations, namely record **updating**. For instance, suppose record

#	**Model**	**Year**	**Color**	**Sale**
5	*Ford*	1991	*Red*	8

cf

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

is updated to

#	**Model**	**Year**	**Color**	**Sale**
5	*Chevy*	1991	*Red*	8

cf

t'_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	1	0
Ford	0	0	1	1	0	1

Properties of data cubing

One just has to compute the "delta" projection,

$$\delta_{Model} = t'_{Model} - t_{Model} =$$

	1	2	3	4	5	6
Chevy	0	0	0	0	1	0
Ford	0	0	0	0	-1	0

then the "delta cube",

$$d = (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v'$$

where

$$v' = (t_{Year} \triangledown (t_{Color} \triangledown \delta_{Model})) \cdot (t^{Sale})^{\circ}$$

and finally add the "delta cube" to the original cube:

$$c' = c + d.$$

Properties of data cubing

Cube commutes with vectorization:

Let $X \xleftarrow{M} Y \times C$ *and* $Y \times X \xleftarrow{\mathbf{vec}\, M} C$ *be its* Y*-vectorization. Then*

$$\mathbf{vec}\,(\mathbf{cube}\;M) = \mathbf{cube}\;(\mathbf{vec}\,M) \tag{15}$$

holds. $\square$

Type diagrams:

$$\begin{tikzcd}
Y \times X \arrow[d, "\tau_Y \otimes \tau_M"'] & & C \arrow[ll, "\mathbf{vec}_Y\; M"'] & \cong & X \arrow[d, "\tau_X"'] & & Y \times C \arrow[ll, "M"'] \\
(Y+1) \times (X+1) & & C+1 \arrow[ll, shift right, "\mathbf{cube}\,(\mathbf{vec}_Y\; M)"'] \arrow[ll, shift left, "\mathbf{vec}_{Y+1}\,(\mathbf{cube}\;\; M)"] \arrow[u, "\tau_C^\circ"'] & \cong & X+1 & & (Y+1) \times (C+1) \arrow[ll, "\mathbf{cube}\; M"'] \arrow[u, "(\tau_Y \otimes \tau_C)^\circ"']
\end{tikzcd}$$

(Proof in the paper.)

Properties of data cubing

The following theorem shows that changing the dimensions of a data cube does not change its totals.

Theorem (Free theorem)

Let $B \xleftarrow{M} A$ *be cubed into* $B+1 \xleftarrow{\mathbf{cube}\ M} A+1$, *and* $r : C \to A$ *and* $s : D \to B$ *be arbitrary functions. Then*

$$\mathbf{cube}\ (s^\circ \cdot M \cdot r) \;=\; (s^\circ \oplus id) \cdot (\mathbf{cube}\ M) \cdot (r \oplus id) \qquad (16)$$

holds, where $M \oplus N = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ *is matrix* **direct sum**.

□

The proof given in the paper resorts to the **free theorem** of polymorphic operators popularized by Wadler (1989) under the heading *Theorems for free!*.

Cube universality — slicing

Slicing is a specialized filter for a particular value in a dimension.

Suppose that from our starting cube

$$c : 1 \rightarrow (Year + 1) \times ((Color + 1) \times (Model + 1))$$

one is only interested in the data concerning year 1991.

It suffices to regard data values as (categorial) **points**: given $p \in A$, constant function $\underline{p} : 1 \rightarrow A$ is said to be a *point* of A, for instance

$$\underline{1991} : 1 \rightarrow Year + 1$$

$$\underline{1991} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Cube universality — slicing

Example:

$$1 \xrightarrow{\mathit{c}} (\mathit{Year}+1)\times((\mathit{Color}+1)\times(\mathit{Model}+1)) \xrightarrow{\underline{1991}^{\circ}\otimes \mathit{id}} 1\times((\mathit{Color}+1)\times(\mathit{Model}+1)) = \begin{bmatrix} 0 \\ 7 \\ 7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 8 \\ 0 \\ 15 \\ 15 \end{bmatrix}$$

Cube universality — rolling-up

Gray et al. (1997) say that *going up the levels [of aggregated data] is called rolling-up*. In this sense, a **roll-up** operation over dimensions A, B and C could be the following form of (increasing) summarization:

$$A \times (B \times C)$$
$$A \times B$$
$$A$$
$$1$$

How does this work over a data cube? We take the simpler case of two dimensions A, B as example.

Cube universality — rolling-up

The dimension powerset for A, B is captured by the corresponding matrix **injections** onto the cube target type $(A+1) \times (B+1)$:

$$
\begin{tikzcd}
 & (A+1) \times (B+1) & & \\
A \times B \arrow[ur, "\theta"] & A \arrow[u, "\alpha"] & B \arrow[ul, "\beta"] & 1 \arrow[ull, "\omega"']
\end{tikzcd}
$$

where

$$
\begin{aligned}
\theta &= i_1 \otimes i_1 \\
\alpha &= i_1 \triangledown i_2 \cdot ! \\
\beta &= i_1 \cdot ! \triangledown i_2 \\
\omega &= i_2 \triangledown i_2
\end{aligned}
$$

NB: the injections i_1 and i_2 are such that $[i_1|i_2] = id$, where $[M|N]$ denotes the horizonal gluing of two matrices.

Cube universality — rolling-up

One can build compound injections, for instance

$$\rho : (A+1) \times (B+1) \leftarrow A \times B + (A+1)$$
$$\rho = [\theta | [\alpha | \omega]]$$

Then, for $M : C \to A \times B$:

$$\rho^\circ \cdot (\textbf{cube}\ M) = \left[\frac{M}{\left[\frac{fst \cdot M}{! \cdot M}\right]} \right] \cdot \tau_C^\circ$$

extracts from $\textbf{cube}\ M$ the corresponding **roll-up**.

The next slides give a concrete example.

Cube universality — rolling-up

Let M be the (generalized) data cube

		1990	1991	ALL
	Chevy	87	0	87
Blue	*Ford*	99	7	106
	ALL	186	7	193
	Chevy	0	0	0
Green	*Ford*	64	0	64
	ALL	64	0	64
	Chevy	5	0	5
Red	*Ford*	0	8	8
	ALL	5	8	13
	Chevy	92	0	92
ALL	*Ford*	163	15	178
	ALL	255	15	270

Cube universality — rolling-up

Building the injection matrix $\rho = [\theta | [\alpha | \omega]]$ for types $\mathit{Color} \times \mathit{Model} + \mathit{Color} + 1 \to (\mathit{Color} + 1) \times (\mathit{Model} + 1)$ we get the following matrix (already transposed):

		Blue			*Green*			*Red*			ALL		
		Chevy	*Ford*	ALL	*Chevy*	*Ford*	ALL	*Chevy*	*Ford*	ALL	*Chevy*	*Ford*	ALL
Blue	*Chevy*	1	0	0	0	0	0	0	0	0	0	0	0
	Ford	0	1	0	0	0	0	0	0	0	0	0	0
Green	*Chevy*	0	0	0	1	0	0	0	0	0	0	0	0
	Ford	0	0	0	0	1	0	0	0	0	0	0	0
Red	*Chevy*	0	0	0	0	0	0	1	0	0	0	0	0
	Ford	0	0	0	0	0	0	0	1	0	0	0	0
	Blue	0	0	1	0	0	0	0	0	0	0	0	0
	Green	0	0	0	0	0	1	0	0	0	0	0	0
	Red	0	0	0	0	0	0	0	0	1	0	0	0
	ALL	0	0	0	0	0	0	0	0	0	0	0	1

Cube universality — rolling-up

Then

$$\rho^\circ \cdot \mathbf{cube}\ M \ = $$

		1990	1991	ALL
Blue	*Chevy*	87	0	87
	Ford	99	7	106
Green	*Chevy*	0	0	0
	Ford	64	0	64
Red	*Chevy*	5	0	5
	Ford	0	8	8
	Blue	186	7	193
	Green	64	0	64
	Red	5	8	13
	ALL	255	15	270

Note how a roll-up is a particular "subset" of a cube.

Matrix ρ° performs the (quantitative) selection of such a subset.

Summary

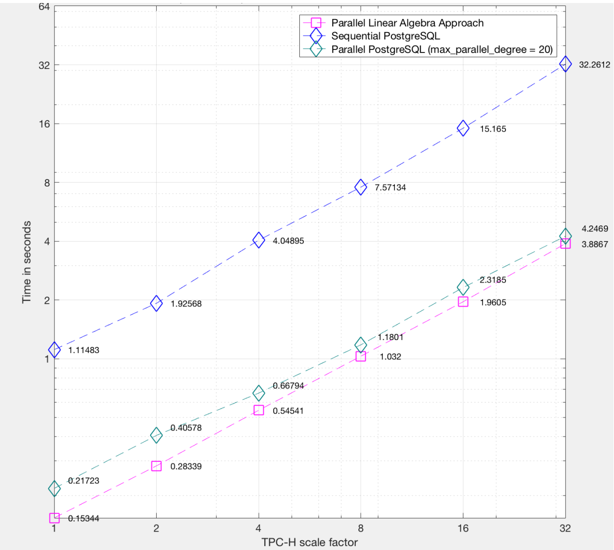
- Abadir and Magnus (2005) stress on the need for a **standardized** notation for **linear algebra** in the field of econometrics and statistics.
- Since (Macedo and Oliveira, 2013) the authors have invested in **typing** linear algebra in a way that makes it closer to modern **typed** languages.
- This talk has shown such a typed approach at work with an example — defining and proving properties of the **data cube** operator.
- This extends previous efforts on applying LA to OLAP (Macedo and Oliveira, 2015)
- Our main aim is to formalize previous work in the field — e.g. by Datta and Thomas (1999) and by Pedersen and Jensen (2001) — in an unified way.

Future work

- We wish to exploit the **parallelism** inherent in linear algebra (LA) processing to implement data cubing in an efficient, parallel way.
- The properties of **cube** can be used to **optimize** LA scripts involving data cubes.
- Preliminary results (Oliveira, 2016; Pontes et al., 2017) show LA scripts encoding data analysis operations performing better on HPC architectures than standard competitors.

Preliminary results (TPC-H on Search6)

(Filipe Oliveira, Sérgio Caldas, MSc project on HPC)



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